Algebraic construction of quantum integrable models including inhomogeneous models

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Abstract

Exploiting the quantum integrability condition we construct an ancestor model associated with a new underlying quadratic algebra. This ancestor model represents an exactly integrable quantum lattice inhomogeneous anisotropic model and at its various realizations and limits can generate a wide range of integrable models. They cover quantum lattice as well as field models associated with the quantum R-matrix of trigonometric type or at the undeformed $q \to 1$ limit similar models belonging to the rational class. The classical limit likewise yields the corresponding classical discrete and field models. Thus along with the generation of known integrable models in a unifying way a new class of inhomogeneous models including variable mass sine-Gordon model, inhomogeneous Toda chain, impure spin chains etc. are constructed.

1 Introduction

Classical integrable systems in $1+1$ and $0+1$ dimensions have been given an unifying picture through possible reductions of the self-dual Yang-Mills equation [1]. However, such success could not be achieved in the quantum case and there is therefore a genuine need for discovering some scheme, which would generate quantum integrable models (QIM) [2] along with their Lax operators and R-matrices in an unifying way.

Such a scheme should be general enough to describe the lattice as well as the field models and quantum as well as the corresponding classical models. Similarly it should also have

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freedom to switch over from relativistic to nonrelativistic and from anisotropic to isotropic models. Therefore it is natural to demand such unifying ancestor model to be a quantum (with quantum parameter $\hbar$), lattice (with lattice constant Δ) model containing a relativistic or anisotropic parameter q along with some possible inhomogeneity parameters $\{c\}$. From such a model therefore we can produce different types of models at different limits of $\Delta, \hbar$, $\{c\}$ and q covering a wide range of integrable models. For example at lattice constant $\Delta \to 0$ one would generate relativistic quantum field models like sine-Gordon, Liouville model etc. and with further limit $q \to 0$ we should get nonrelativistic field models like NLS model. $\Delta \neq 0$ would yield the corresponding lattice versions and also discrete models like relativistic Toda chain, anisotropic XXZ spin chain etc. or at $q \to 1$ limit, the nonrelativistic Toda or the isotropic XXX spin chain. $\hbar \to 0$ at the same time should recover the corresponding classical discrete or field models at appropriate limits. $\{c\} \neq 0$ on the other hand yields a new class of inhomogeneous models.

To seek the unifying structure underlying such a general integrable system, we note that quantum integrable systems exhibit intimate connections with Hopf algebras like Lie algebras and their quantum deformations [4]-[10]. Therefore, motivated by these facts and our earlier experience [11],[12], we construct an ancestor model, which is quantum, discrete, q-deformed and also inhomogeneous. This model is associated with a new Hopf algebra, which however is not introduced by hand but dictated by the integrability condition, i.e. the Yang-Baxter equation itself. Therefore whenever the fields of the constructed model satisfy the underlying algebra, its quantum integrability is guaranteed automatically. The ancestor model itself may be considered as a novel, inhomogeneous, exactly integrable generalized quantum lattice sine-Gordon model and turns out to be an excellent candidate for generating a wide range of integrable quantum models with 2×2 Lax operators. The associated R-matrix is either the known trigonometric one or its corresponding rational form.

Thus our scheme unifies a large class of integrable quantum models by generating them in a systematic way through reductions of an ancestor model with explicit Lax operator realization. Note that the Lax operator together with the quantum R-matrix define an integrable system completely, giving also all conserved quantities including the Hamiltonian of the model. To stress on the wide varieties of the well known quantum integrable models mentioned above, we have provided a short list of them along with their representative Lax operators in the Appendix. This would be helpful for the ready reference and to follow their derivation from the single ancestor model in our scheme.

2 The unifying algebra

The unifying algebra proposed in our scheme is found to be a Hopf algebra. It is more general than the well known quantum Lie algebra and in contrast represents a quadratic algebra (QdA), so called because the generators in the defining algebraic relations appear in the quadratic form. The algebra may be defined by the relations

$$[S^3, S^{\pm}] = \pm S^{\pm}, \quad [S^+, S^-] = \Big(M^+ \sin(2\alpha S^3) + M^- \cos(2\alpha S^3)\Big) \frac{1}{\sin\alpha}, \quad [M^{\pm}, \cdot] = 0, \tag{1}$$

where $M^{\pm}$ are the central elements. We show that (1) is not merely a modification of the known $U_q(su(2))$ but is a QdA underlying an integrable ancestor model and is a consequence of the quantum Yang Baxter equation (QYBE)

$$R(\lambda-\mu)L(\lambda)\otimes L(\mu) = (I\otimes L(\mu))\otimes(L(\lambda)\otimes I)R(\lambda-\mu). \tag{2}$$

We take the associated 4×4 quantum $R(\lambda)$-matrix as the well known solution related to the sine-Gordon model, with its nontrivial elements given by [2]

$$R^{11}_{11} = R^{22}_{22} = a(\lambda),\ R^{12}_{12} = R^{21}_{21} = b(\lambda),\ R^{12}_{21} = R^{21}_{12} = c, \tag{3}$$

and expressed through trigonometric functions of spectral parameters as

$$a(\lambda) = \sin(\lambda+\alpha), \quad b(\lambda) = \sin\lambda, \quad c = \sin\alpha. \tag{4}$$

On the other hand, we choose the Lax operator of our ancestor model as

$$L^{(anc)}_t(\xi) = \begin{pmatrix} \xi c_1^+ e^{i\alpha S^3} + \xi^{-1} c_1^- e^{-i\alpha S^3} & 2\sin\alpha S^- \\ 2\sin\alpha S^+ & \xi c_2^+ e^{-i\alpha S^3} + \xi^{-1} c_2^- e^{i\alpha S^3} \end{pmatrix}, \quad \xi = e^{i\alpha\lambda}. \tag{5}$$

with $c_a^{\pm}$ being central to the algebra (1) through the relation $M^{\pm} = \pm\sqrt{\pm 1}(c_1^+ c_2^- \pm c_1^- c_2^+)$. The derivation of algebra (1) follows directly from QYBE by inserting in it the explicit forms of the Lax operator (5) and the R-matrix (3) and matching different powers of the spectral parameter ξ. Therefore algebra (1) or its various realizations in effect becomes equivalent to the QYBE and guarantees the quantum integrability of the model constructed on it. To establish that (1) is a Hopf algebra we show that the following characteristics known as the coproduct $\Delta(x) : A \to A\otimes A$, antipode $S : A \to A$ and the counit $\epsilon : A \to k$, along with a multiplication $M : A\otimes A \to A$ and a unit $\eta : k \to A$ hold for algebra (1). All these taken together defines it as a Hopf algebra. For deriving these objects we exploit another inherent property of the QYBE, namely the product of two Lax operators $L_{aj}L_{aj+1}$ as well as L^{-1}_{aj} represent also solutions of QYBE indicating its inherent Hopf algebra structure. Using Lax operator (5) therefore we can derive the coproducts in the explicit form

$$\begin{gathered} \Delta(S^+) = c_1^+ e^{i\alpha S^3}\otimes S^+ + S^+\otimes c_2^+ e^{-i\alpha S^3} \quad , \quad \Delta(S^-) = c_2^- e^{i\alpha S^3}\otimes S^- + S^-\otimes c_1^- e^{-i\alpha S^3} \\ \Delta(S^3) = I\otimes S^3 + S^3\otimes I \quad , \quad \Delta(c_i^{\pm}) = c_i^{\pm}\otimes c_i^{\pm}. \end{gathered} \tag{6}$$

Inserting the coproducts in the algebraic relations (1) one can prove after some easy steps that the same algebra is also true for its tensor product expressed through (6). The Hopf algebra property is a key for obtaining the same YBE relations again for the global object

$T(\lambda) = \prod_{i=1}^N L_i(\lambda)$: $\quad R_{12}(\lambda-\mu)\ T(\lambda)\ \otimes T(\mu) = (I \otimes T(\mu)\ \ T \otimes I(\lambda)\ \ R_{12}(\lambda-\mu)$. This in turn derives for $t(\lambda) = trT(\lambda)$ the quantum integrability condition $[t(\lambda), t(\mu)] = 0$.

Similarly one may derive the antipode or the 'inverse' for the algebra as

$$S(S^-) = -(c_1^+)^{-1} e^{-i\alpha S^3} S^- e^{i\alpha S^3} (c_2^+)^{-1}, \quad S(S^+) = -(c_2^-)^{-1} e^{-i\alpha S^3} S^+ e^{i\alpha S^3} (c_1^-)^{-1},$$

$$S(c_i^\pm) = (c_i^\pm)^{-1}, \quad S(e^{\pm i\alpha S^3}) = e^{\mp i\alpha S^3},$$

which satisfy also the same algebraic relations (1). The counit $\epsilon(c_i^\pm) = 1, \quad \epsilon(e^{\pm i\alpha S^3}) = 1, \ \epsilon(S^\pm) = 0$ on the other hand maps the algebra to some number identities. For multiplication M one can take the formal definition in the algebra, while the unit η may be defined through the unital element 1 as $\eta(\xi) \to \xi 1$.

We also observe that unlike Lie algebras or their deformations, due to the presence of multiplicative operators $M^\pm$ in (1), it represents quantum-deformation of a QdA. Since these operators have arbitrary eigenvalues including zeros, they can not be removed by scaling and therefore generically (1) is different from the known quantum algebra. Moreover different representations of M's generate new structure constants leading to a rich variety of deformed Lie algebras, which are related to different integrable systems. This fact becomes important for its present application. The appearance of QdA in the basic integrable system should be rather expected, since the QYBE with R-matrix having c-number elements is itself a QdA. The notion of QdA was introduced first by Sklyanin [13].

3 Generation of models

For constructing physical models we have to find first representations of (1) in physical variables like canonical variables u, p with commutation relations $[u, p] = i$, or bosonic operators $\psi, \psi^\dagger$ commuting as $[\psi, \psi^\dagger] = 1$ or the spin variables $s^\pm, s^3$. One such representations may be given by

$$S^3 = u, \quad S^+ = e^{-ip} g(u), \quad S^- = g(u) e^{ip}. \tag{7}$$

where the operator function

$$g(u) = \Big(\kappa + \sin\alpha(s-u)(M^+ \sin\alpha(u+s+1) + M^- \cos\alpha(u+s+1))\Big)^{\frac{1}{2}} \frac{1}{\sin\alpha} \tag{8}$$

containing free parameters κ and s. Inserting this realization in the ancestor Lax operator (5) one gets a novel exactly integrable quantum model *generalizing lattice SG model*. It is evident that only for hermitian $g(u)$ one gets $S^- = (S^+)^\dagger$. We show below the remarkable fact that by fixing different eigenvalues of $M^\pm$ in the same form (8) of $g(u)$ and taking different limits of the parameters involved and at the same time choosing various realizations we can derive a whole range of quantum integrable models including new models. As an added advantage, the Lax operators of the models are obtained automatically from (5), while the R-matrix is simply inherited. The underlying algebras of the models are also given by the corresponding representations of the ancestor algebra (1). One of the reasons for the success of this scheme is the quadratic nature of the algebra (1) with the explicit appearance of Casimir operators $M^{\pm}$. Using this feature It is possible to build a new class of models, that may be considered as the inhomogeneous versions of the existing integrable models. The idea of such construction is to take locally different representations for the central elements, i.e. instead of taking their eigenvalues as constants one should consider them to be site (and time) dependent functions. This simply means that in the expressions for $g(u_j)$ in (8), $M^{\pm}$ should be considered as $M_j^{\pm}$ and consequently in Lax operator (5) all c's should be lattice indexed as c_j's. Thus the values of central elements may vary arbitrarily at different lattice points leading to inhomogeneous lattice models. However since the algebra remains the same they answer to the same quantum R-matrices. Physically such inhomogeneities may be interpreted as impurities, varying external fields, incommensuration etc.

3.1 Models belonging to trigonometric class

Before constructing inhomogeneous models we show first that the known models, which were discovered earlier in isolation and mainly through quantization of the existing classical models, can be reproduced in a unified way from our single ancestor model.The equations corresponding to such models along with the explicit forms of their Lax operators are listed in the Appendix for comparison. For the eigenvalues of the Casimir operators fixed at $M^- = 0, M^+ = 1$, as easily seen, (1) reduces to the well known quantum algebra $U_q(su(2))$ [3] given by

$$[S^3, S^{\pm}] = \pm S^{\pm}, [S^+, S^-] = [2S^3]_q. \tag{9}$$

Now the simplest representation $\vec{S} = \frac{1}{2}\vec{\sigma}$ derives the integrable *XXZ spin chain* [5], from (5). On the other hand, representation (7) with the corresponding reduction of (8) as $g(u) = \frac{1}{2\sin\alpha}\left[1+\cos\alpha(2u+1)\right]^{\frac{1}{2}}$ recovers the quantum exact *lattice sine-Gordon* model [14] with its Lax operator obtained directly from (5) with all c's $= 1$ (which is compatible with $M^- = 0, M^+ = 1$).

Another unusual Lie algebra can be generated from (1) by fixing the eigenvalues of c's as $c_1^+ = c_2^- = 1, \quad c_1^- = c_2^+ = 0$ which correspond to the values $M^{\pm} = \pm\sqrt{\pm 1}$. This gives an exponentially deformed Lie algebra

$$[S^+, S^-] = \frac{e^{2i\alpha S^3}}{2i\sin\alpha}. \tag{10}$$

and reduces (8) to $g(u) = \frac{(1+e^{i\alpha(2u+1)})^{\frac{1}{2}}}{\sqrt{2}\sin\alpha}$. The Lax operator (5) with these values of c's and the explicit form of $g(u)$ clearly yields the *exact lattice version of the quantum Liouville* model [15]. Note that the present values of $M^{\pm}$ may be achieved even with $c_1^- \neq 0$, giving the same algebra (since (1) depends only on $M^{\pm}$) and hence the same realization. However, the Lax operator which depends explicitly on c's gets changed reducing (5) to another nontrivial structure. This is an interesting possibility of constructing systematically different useful Lax operators for the same model. For example, the present construction of the *second Liouville Lax operator* recovers easily that of [6], invented in an involved way for its Bethe ansatz solution. On the other hand for a bit asymmetric choice of eigenvalues: $c_1^+ = c_2^+ = 1,\ c_1^- = -iq,\ c_2^- = \frac{i}{q}$ leading to $M^+ = 2\sin\alpha,\ M^- = 2i\cos\alpha$ we may use another realization

$$S^+ = -\kappa A,\ S^- = \kappa A^\dagger,\ S^3 = -N, \quad \kappa = -i(\cot\alpha)^{\frac{1}{2}}, \tag{11}$$

which reduces (1) directly to the well known q-oscillator algebra [16, 17]:

$$[A, N] = A, \quad [A^\dagger, N] = -A^\dagger, \quad [A, A^\dagger] = \frac{\cos(\alpha(2N+1))}{\cos\alpha}. \tag{12}$$

Note that such direct derivation of q-oscillator algebra is not possible to obtain from the quantum algebra $U_q(su(2))$. Another interesting point is that, the coproduct structure (6) of the Casimir operators $\Delta(c_i^\pm) = c_i^\pm \otimes c_i^\pm$ clearly contradicts the present values of $c_a^-, a = 1, 2$, which leads to the q-oscillator algebra (12). This provides perhaps an unexpected argument why q-oscillator algebra lacks the Hopf algebra structure. Thus we have constructed a new quantum integrable *q-oscillator model*, from which a more physical model can also be obtained. Using the realization (7) with (8) simplified as $g^2(u) = [-2u]_q$ and the mapping to the bosonic operators: $\psi = e^{-ip}((s-u))^{\frac{1}{2}}, N = s - u$, with $[\psi, \psi^\dagger] = 1$, we can express the q-oscillator through standard bosons as $A = \ \psi(\frac{[2N]_q}{2N\cos\alpha})^{\frac{1}{2}},\ N = \psi^\dagger\psi$. This converts the integrable q-oscillator model into a bosonic model representing an exact lattice version of the quantum *derivative nonlinear Schrödinger equation* (QDNLS) [11] with (5) yielding the corresponding Lax operator. The QDNLS was shown to be related to the interacting bose gas with derivative δ-function potential [18]. Fusing two such models one can further create an integrable *massive Thirring* model described in [2].

Since our quadratic algebra allows also trivial eigenvalues for $M^\pm$, we may choose even $M^\pm = 0$. Note that this case may be achieved with different sets of eigenvalues for c as *i)* $c_a^+ = 1$, $a = 1, 2$, or *ii)* $c_1^\mp = \pm 1$, or *iii)* $c_1^+ = 1$, with the rest of c's being zeros. All these sets of values lead to the same underlying algebra

$$[S^+, S^-] = 0,\ [S^3, S^\pm] = \pm S^\pm. \tag{13}$$

However, they may generate different Lax operators from (5) due to its explicit dependence on c. In particular, case i) leads to the *light-cone SG* model, while ii) and iii) give two different Lax operators found in [19] and [20] for the same relativistic Toda chain. We have also seen above such examples of constructing different convenient Lax operators for the same Liouville model. In the present case (8) gives simply $g(u)$ =const., therfore interchanging canonically $u \to -ip, p \to -iu$, (7) yields

$$S^3 = -ip,\ S^\pm = \alpha e^{\mp u} \tag{14}$$

and the ancestor Lax operator generates evidently the *discrete-time or relativistic quantum Toda chain.*

3.2 Models belonging to the rational class

For covering a wide range of models trough reductions of our ancestor model various free parameters are inbuilt in it. One of them is the deformation parameter $q = e^{\alpha}$, which was kept generic for the above trigonometric class. Now we consider the undeformed limit $q \to 1$ or equivalently $\alpha \to 0$ for generating the isotropic or the nonrelativistic models belonging to the rational class and examine how the structure of the main objects in our scheme gets suitably modified. It is immediate that for the existence of such a limit the central elements along with the generators must be α dependent. A consistent procedure leads to $S^{\pm} \to is^{\pm}, M^{+} \to -m^{+}, M^{-} \to -\alpha m^{-}, \xi \to 1 + i\lambda$ reducing (1) to the algebraic relations

$$[s^+, s^-] = 2m^+ s^3 + m^-, \qquad [s^3, s^{\pm}] = \pm s^{\pm} \tag{15}$$

with $m^+ = c_1^0 c_2^0, \quad m^- = c_1^1 c_2^0 + c_1^0 c_2^1$ as the new central elements. We see again that it is not a Lie but a QdA, since the multiplicative operators $m^{\pm}$ can not be removed in general due to their allowed zero eigenvalues. Moreover, algebra (15) is also a Hopf algebra with the coproduct structure

$$\begin{aligned} \Delta(s^+) &= c_1^0 \otimes s^+ + s^+ \otimes c_2^0, \quad \Delta(s^-) = c_2^0 \otimes s^- + s^- \otimes c_1^0, \quad \Delta(s^3) = I \otimes s^3 + s^3 \otimes I \\ \Delta(c_i^0) &= c_i^0 \otimes c_i^0, \quad \Delta(c_i^1) = c_i^0 \otimes c_i^1 + c_i^1 \otimes c_i^0 \end{aligned} \tag{16}$$

giving $\Delta(m^+) = m^+ \otimes m^+, \Delta(m^-) = m^+ \otimes m^- + m^- \otimes m^+$. An unusual feature of (15) can be observed from (17), that though being an undeformed algebra it is noncocommutative in nature.

The ancestor Lax operator (5) at the limit $\alpha \to 0$ is converted naturally into

$$L_r(\lambda) = \begin{pmatrix} c_1^0(\lambda + s^3) + c_1^1 & s^- \\ s^+ & c_2^0(\lambda - s^3) - c_2^1 \end{pmatrix}, \tag{17}$$

with rational dependence on spectral parameter λ and the quantum R-matrix (3) is reduced also to its rational form with

$$a(\lambda) = \lambda + \alpha, \quad b(\lambda) = \lambda, \quad c = \alpha, \tag{18}$$

well known for the NLS model [2]. Therefore the integrable systems associated with algebra (15) and generated by ancestor model (17) would belong now to a different class, namely the rational class, all sharing the same rational R-matrix (18). The corresponding models, a few examples of which are listed in the appendix, are of nonrelativistic or the isotropic type.

It is interesting to find that at $\alpha \to 0$ the operator function (8) after putting $\kappa = 0$ gives $g_0(u) = i((s-u)(m^+(u+s+1) + m^-))^{\frac{1}{2}}$, which using the inter-bosonic map reduces the representation (7) into a generalized Holstein-Primakov transformation (HPT)

$$s^3 = s - N, \quad s^+ = g_0(N)\psi, \quad s^- = \psi^{\dagger} g_0(N), \quad g_0^2(N) = m^- + m^+(2s - N), \quad N = \psi^{\dagger}\psi. \tag{19}$$

It can be checked to be an exact realization of (15) associated with the Lax operator (17) and therefore may be considered as an integrable *generalized lattice NLS* model. This may serve now as an ancestor model for the rational class. For generating first the standard homogeneous models we choose constant eigenvalues for the Casimir operators. One such choice $m^+ = 1, m^- = 0$, reduces (15) clearly to the $su(2)$ algebra, which for spin $\frac{1}{2}$ representation gives the XXX *spin chain* [21] from the Lax operator (17). Bosonic realization (19) in this case simplifies to the standard HPT with $g_0^2(N) = (2s - \psi^\dagger \psi)$, reproducing the known *lattice NLS* model [14] from (17).

A complementary choice $m^+ = 0, m^- = 1$, on the other hand converts (19) simply to $s^+ = \psi, s^- = \psi^\dagger, s^3 = s - N$ due to $g_0(N) = 1$ and reduces (15) directly to the oscillator algebra

$$[\psi, N] = \psi, \quad [\psi^\dagger, N] = -\psi^\dagger, \qquad [\psi, \psi^\dagger] = 1. \tag{20}$$

(17) with this realization generates another *simple lattice NLS* model [22]. Remarkably, the trivial choice $m^\pm = 0$ gives again algebra (13) and therefore the same realization (14) found for the relativistic case can also be used for the *nonrelativistic Toda chain* [2]. The related Lax operator associated with the rational R-matrix, however should be obtained from (17). It should be noted that a bosonic realization of general Lax operators like (5) and (17) can be found also in some earlier works [4, 23].

4 Field models and classical models

As we required, our ancestor model apart from the discrete quantum systems obtained above, is capable also of constructing the integrable family of quantum field models as well as classical discrete and field models at different limits. For obtaining the quantum field models we have to start from their respective lattice versions constructed above and scale the lattice operators $p_j, u_j, c_a^\pm, \psi_j$, consistently by the lattice spacing Δ. At the continuum limit $\Delta \to 0$ one would obtain the field operators $p_j \to p(x), \psi_j \to \psi(x)$ etc. along with the commutation relations like $[\psi_j, \psi_k^\dagger] = \frac{\delta_{jk}}{\Delta} \;\to\; [\psi(x), \psi(y)] = \delta(x-y)$. The corresponding Lax operator $\mathcal{L}(x, \lambda)$ for the continuum model is obtained from its discrete counterpart as $L_j(\lambda) \to I + i\Delta\mathcal{L}(x, \lambda) + O(\Delta^2)$. The associated R-matrix however remains the same, since it does not contain lattice constant Δ. Thus *integrable field models* like sine-Gordon, Liouville, NLS or the derivative NLS models are obtained (see appendix) from their discrete variants constructed above.

At the classical limit $\hbar \to 0$, all the field operators are converted into ordinary functions with their commutators reducing to the Poisson brackets. Note that the $\hbar$ enters the R-matrix as the scaling factor $\hbar\alpha$ into its elements, which in turn determine the structure constants of the algebraic relations. Therefore the ancestor algebras (1) and (15) are converted into the Poisson algebras and the R-matrix to its classical form: $R(\lambda) = I + \hbar(\lambda) + O(\hbar^2)$. However our basic scheme remains almost the same. Remarkably, the quantum parameter does not enter the Lax operators explicitly therefore the form of the ancestor Lax operators (5) and

(17) remains in the same form also in the classical limit. Therefore we can use all the above reductions of the ancestor models for generating the classical model with the same success.

5 Inhomogeneous models

As mentioned above the ancestor model (5) and its undeformed variant (17) containing nontrivial Casimir operators can be used for constructing a new class of integrable inhomogeneous models. For this the eigenvalues of the Casimir operators should be chosen as site dependent, and in general time dependent functions.

Notice that in the sine-Gordon model unlike its coupling constant the mass parameter enters through the Casimir operator of the underlying algebra. Therefore taking $M_j^+ = -(\Delta m_j)^2$, one can construct a variable mass (in general time dependent) discrete SG model without spoiling its integrability. In the continuum limit it would result a novel *variable mass sine-Gordon field model* with the Hamiltonian

$$\mathcal{H} = \int dx \left[m(x,t)(u_t)^2 + (1/m(x,t))(u_x)^2 + 8(m_0 - m(x,t)\cos(2\alpha u)) \right], \tag{21}$$

which may arise also in physical situations [24].

Inhomogeneous lattice NLS model can be obtained by considering site-dependent values for central elements in (17) and in the generalized HPT (19). As a possible quantum field model it may correspond to equations like *cylindrical NLS* [25] with explicit coordinate dependent coefficients. In a similar way inhomogeneous versions of Liouville model, (non)relativistic Toda etc. can be constructed. For example, taking $c_1^a \to c_j^a$ in nonrelativistic Toda chain we can get a new integrable *inhomogeneous quantum Toda chain* with the Hamiltonian

$$H = \sum_j (p_j + \frac{c_j^1}{c_j^0})^2 + \frac{1}{c_j^0 c_{j+1}^0} e^{u_j - u_{j+1}}. \tag{22}$$

Note that such inhomogeneities can not be removed through gauge transformation or variable change, if the inhomogeneity c_j^a are time-dependent functions.

Another way of constructing inhomogeneous models is to use different realizations of algebras (1) or (15) at different lattice sites, depending on the type of R-matrix. This may lead even to different underlying algebras and hence different Lax operators at differing sites opening up possibilities of building various exotic inhomogeneous models. For example, it may be possible to construct a hybrid models of sine-Gordon and Liouville fields, where for $x \geq 0$ it would follow the sine-Gordon dynamics, while for $x < 0$ the Liouville dynamics. Such different possibilities of model constructions will be dealt elsewhere.

Similarly nontrivial examples of impurity XXZ (or XXX) spin chains can be obtained, if we replace its standard Lax operator at a single impurity site m by the ancestor model (5) (or (17)) itself or any of its representations (in spin, q-oscillator or boson). This would give rise to integrable quantum spin chains *with impurity* of various nature, which might have physical significance. Such a spin Hamiltonian with impurity would look like

$$H = -(\sum_{j \neq m, m-1} L^{xxx}(0)'_{jj+1} (L^{xxx}(0))^{-1}_{jj+1} + L^d(0)'_{mm+1} L^b(d))^{-1}_{mm+1}$$

$$+ L^d(0)_{mm-1} (L^{xxx}(0)'_{m+1m-1} L^{xxx}(0)^{-1}_{m+1m-1}) L^d(0)^{-1}_{mm-1})$$

where $L^{xxx}(\lambda)$ and $L^d(\lambda)$ correspond to the spin and the impurity Lax operators, respectively. Such a Lax operator with bosonic impurity may be derived from (17) as

$$L^b_m(\lambda) = \begin{pmatrix} (\lambda - N_m) + \frac{3}{2} & a^\dagger_m \\ a_m & -1 \end{pmatrix}. \tag{23}$$

6 Concluding remarks

Thus we have prescribed an unifying scheme for constructing integrable systems, which covers quantum lattice as well as field models of both relativistic or anisotropic (with $q \neq 1$) and nonrelativistic or isotropic models (with $q = 1$) along with their corresponding classical counterparts. Such models can be generated systematically from a single ancestor model with underlying algebra (1). In general using the freedom of choosing the eigenvalues of the Casimir operators appearing in the algebra to be site as well as time dependent functions, we obtain inhomogeneous quantum integrable models constituting a new class. The Lax operators of the descendant models are constructed from (5) or its $q \to 1$ limit (17), while the variety of their concrete representations are obtained from the same general form (7) at different realizations. The corresponding underlying algebraic structures are the allowed reductions of (1). The associated quantum R-matrix of the descendant models however remains the same trigonometric or the rational form as inherited from the ancestor model. This answers to the mystery why a wide range of integrable models are found to share the same R-matrices. This fact also shows an intimate relationship among the descendant models inspite of their seemingly wide external differences and reveals a universal character for solving these models through algebraic Bethe ansatz (ABA) [2, 26], where the elements of the associated R-matrix plays the major role.

7 Appendix: Well known examples of quantum integrable models and their Lax operators

7.1 Models associated with trigonometric R-matrix

i) *Field models*

1. Sine-Gordon model

$$u_{tt} - u_{xx} = \frac{m^2}{\eta} \sin(\eta u), \qquad \mathcal{L} = \begin{pmatrix} ip, & m \sin(\lambda - \eta u) \\ m \sin(\lambda + \eta u), & -ip \end{pmatrix}. \tag{1}$$

2. Liouville model

$$u_{tt} - u_{xx} = \frac{1}{2} e^{2\eta u}, \qquad \mathcal{L} = i \begin{pmatrix} p, & \xi e^{\eta u} \\ \frac{1}{\xi} e^{\eta u}, & -p \end{pmatrix}. \tag{2}$$

3. A derivative NLS (DNLS) model

$$i\psi_t - \psi_{xx} + 4i(\psi^\dagger \psi \psi_x = 0, , \quad \mathcal{L} = i \begin{pmatrix} -\frac{1}{4}\xi^2 + k_- \psi^\dagger \psi, & \xi \psi^\dagger \\ \xi \psi, & \frac{1}{4}\xi^2 - k_+ \psi^\dagger \psi \end{pmatrix}. \tag{3}$$

ii) *Lattice Models*

1. Anisotropic XXZ spin chain

$$\mathcal{H} = \sum_n^N \sigma_n^1 \sigma_{n+1}^1 + \sigma_n^2 \sigma_{n+1}^2 + \cos\eta \sigma_n^3 \sigma_{n+1}^3, \quad L_n(\xi) = \sin(\lambda + \eta \sigma^3 \sigma_n^3) + \sin\eta \ (\sigma^+ \sigma_n^- + \sigma^- \sigma_n^+) \tag{4}$$

3. Lattice SG model

$$L_n(\xi) = \begin{pmatrix} g(u_n)\ e^{ip_n \Delta}, & m\Delta \sin(\lambda - \eta u_n) \\ m\Delta \sin(\lambda + \eta u_n), & e^{-ip_n \Delta}\ g(u_n) \end{pmatrix}, \quad g(u_n)^2 = 1 + m^2 \Delta^2 \cos\eta(2u_n + 1) \tag{5}$$

4. Lattice Liouville model

$$L_n(\xi) = \begin{pmatrix} e^{p_n \Delta}\ f(u_n)\ , & \Delta \xi e^{\eta u_n} \\ \frac{\Delta}{\xi} e^{\eta u_n}, & f(u_n)\ e^{-p_n \Delta} \end{pmatrix}, \qquad f(u_n)^2 = 1 + \Delta^2 e^{\eta(2u_n + i)} \tag{6}$$

5. Lattice DNLS model (A q-oscillator model)

$$L_n(\xi) = \begin{pmatrix} \frac{1}{\xi} q^{-N_n} - \frac{i\xi\Delta}{4}\ q^{N_n + 1}, & \kappa A_n^\dagger \\ \kappa A_n, & \frac{1}{\xi} q^{N_n} + \frac{i\xi\Delta}{4}\ q^{-(N_n+1)} \end{pmatrix}, \tag{7}$$

7. Relativistic or discrete-time quantum Toda chain

$$H = \sum_i \left(\cosh 2\eta p_i + \eta^2 \cosh \eta (p_i + p_{i+1}) e^{(q_i - q_{i+1})} \right), \quad L_n(\xi) = \begin{pmatrix} \frac{1}{\xi} e^{\eta p_n} - \xi e^{-\eta p_n}, & \eta e^{q_n} \\ -\eta e^{-\eta q_n}, & 0 \end{pmatrix}. \tag{8}$$

7.2 Models associated with rational R-matrix

i)*Field models:*

1. Nonlinear Schrödinger equation (NLS)

$$i\psi_t + \psi_{xx} + \eta^2 (\psi^\dagger \psi) \psi = 0, \quad \mathcal{L}(\lambda) = \begin{pmatrix} \lambda, & \eta \psi \\ \eta \psi^\dagger, & -\lambda \end{pmatrix}. \tag{9}$$

ii) *Lattice Models:*

1. Isotropic XXX spin chain

$$\mathcal{H} = \sum_n^N \sigma_n^1 \sigma_{n+1}^1 + \sigma_n^2 \sigma_{n+1}^2 + \sigma_n^3 \sigma_{n+1}^3, \quad L_n(\lambda) = i(\lambda \mathbf{I} + \eta(\sigma^3 \sigma_n^3 + \sigma^+ \sigma_n^- + \sigma^- \sigma_n^+)) \tag{10}$$

2. Lattice NLS model

$$L_n(\lambda) = \begin{pmatrix} \lambda + s - \Delta\psi^\dagger\psi & \Delta^{\frac{1}{2}}(2s - \Delta\psi^\dagger\psi)^{\frac{1}{2}}\psi^\dagger \\ \Delta^{\frac{1}{2}}\psi(2s - \Delta\psi^\dagger\psi)^{\frac{1}{2}} & \lambda - s + \Delta\psi^\dagger\psi \end{pmatrix}. \tag{11}$$

3. Toda chain (nonrelativistic)

$$H = \sum_i \left(\frac{1}{2}p_i^2 + e^{(q_i - q_{i+1})}\right), \qquad L_n(\lambda) = \begin{pmatrix} p_n - \lambda & e^{q_n} \\ -e^{-q_n} & 0 \end{pmatrix}. \tag{12}$$

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